# Equivariant symmetric bilinear torsions *

Guangxiang Su †

**Abstract**

We extend the main result in the previous paper of Zhang and the author relating the Milnor-Turaev torsion with the complex valued analytic torsion to the equivariant case.

## 1 Introduction

Let $F$ be a unitary flat vector bundle on a closed Riemannian manifold $X$. In [RS], Ray and Singer defined an analytic torsion associated to $(X, F)$ and proved that it does not depend on the Riemannian metric on $X$. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on $X$ (cf. [Mi]). This conjecture was later proved in the celebrated papers of Cheeger [C] and Müller [Mu1]. Müller generalized this result in [Mu2] to the case where $F$ is a unimodular flat vector bundle on $X$. In [BZ1], inspired by the considerations of Quillen [Q], Bismut and Zhang reformulated the above Cheeger-Müller theorem as an equality between the Reidemeister and Ray-Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundles over $X$. The method used in [BZ1] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [W] on the de Rham complex.

On the other hand, Turaev generalizes the concept of Reidemeister torsion to a complex valued invariant whose absolute value provides the original Reidemeister torsion, with the help of the so-called Euler structure (cf. [T], [FT]). It is natural to ask whether there exists an analytic interpretation of this Turaev torsion.

Recently, Burghelea and Haller [BH1, BH2], following a suggestion of Müller, define a generalized analytic torsion associated to a nondegenerate symmetric bilinear form on a flat vector bundle over a closed manifold and make an explicit conjecture between this generalized analytic torsion and the Turaev torsion. Later this conjecture was proved by Su and Zhang [SZ]. Also Burghelea and Haller [BH3], up to a sign, proved this conjecture for odd dimensional manifolds, and comments were made how to derive the conjecture in full generality in their paper.

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†Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China. (sugx@mail.nankai.edu.cn)

In this paper, we will extend the main result in [SZ] to the equivariant case, which is closer in spirit to the approach developed by Bismut-Zhang in [BZ2].

The rest of this paper is organized as follows. In Section 2, we construct the equivariant symmetric bilinear torsions associated with equivariant nondegenerate symmetric bilinear forms on a flat vector bundle. In Section 3, we state the main result of this paper. In Section 4, we provide a proof of the main result. Section 5 is devoted to the proofs of the intermediary results stated in Section 4.

Since we will make substantial use of the results in [BZ1, BZ2, SZ], we will refer to [BZ1, BZ2, SZ] for related definitions and notations directly when there will be no confusion.

# 2 Equivariant symmetric bilinear torsions associated to the de Rham and Thom-Smale complexes

In this section, for a $G$-invariant nondegenerate bilinear symmetric form on a complex flat vector bundle over an oriented closed manifold, we define two naturally associated equivariant symmetric bilinear forms on the equivariant determinant of the cohomology $H^*(M,F)$ with coefficient $F$. One constructed in a combinatorial way through the equivariant Thom-Smale complex associated to a equivariant Morse function, and the other one constructed in an analytic way through the equivariant de Rham complex.

## 2.1 Equivariant symmetric bilinear torsion of a finite dimensional complex

Let $(C,\partial)$ be a finite cochain complex

$$(C,\partial):0\longrightarrow C^0\xrightarrow{\partial_0}C^1\xrightarrow{\partial_1}\cdots\xrightarrow{\partial_{n-1}}C^n\longrightarrow 0, \tag{2.1}$$

where each $C^i$, $0\le i\le n$, is a finite dimensional complex vector space.

Let each $C^i$, $0\le i\le n$, admit a nondegenerate symmetric bilinear form $b_i$. We equip $C$ with the nondegenerate symmetric bilinear form $b_C=\bigoplus_{i=0}^n b_i$.

Let $G$ be a compact group. Let $\rho:G\to \mathrm{End}(C)$ be a representation of $G$, with values in the chain homomorphisms of $C$ which preserve the bilinear form $b_C$. In particular, if $g\in G$, $\rho(g)$ preserves the $C^i$'s.

Let $\widehat{G}$ be the set of equivalence classes of complex irreducible representations of $G$. An element of $\widehat{G}$ is specified by a complex finite dimensional vector space $W$ together with an irreducible representation $\rho_W:G\to\mathrm{End}(W)$.

For $W\in\widehat{G}$, set

$$C^i_W=\mathrm{Hom}_G(W,C^i)\otimes W, \tag{2.2}$$

$$C_W=\mathrm{Hom}_G(W,C)\otimes W. \tag{2.3}$$

Let $\partial_W$ be the map induced by $\partial$ on $C_W$. Then

$$(2.4)\qquad (C_W,\partial_W): 0 \longrightarrow C_W^0 \xrightarrow{\partial_{0,W}} C_W^1 \xrightarrow{\partial_{1,W}} \cdots \xrightarrow{\partial_{n-1,W}} C_W^n \longrightarrow 0$$

is a chain complex. Thus we obtain the isotypical decomposition,

$$(2.5)\qquad (C,\partial) = \bigoplus_{W\in\widehat{G}} (C_W,\partial_W),$$

and the decomposition (2.5) is orthogonal.

If $E$ is a complex finite dimensional representation space for $G$, let $\chi(E)$ be the character of the representation. Put

$$\chi(C) = \sum_{i=0}^{n} (-1)^i \chi(C^i),$$

$$e(C) = \sum_{i=0}^{n} (-1)^i \mathrm{dim} C^i,$$

$$(2.6)\qquad e(C_W) = \sum_{i=0}^{n} (-1)^i \mathrm{dim}(C_W^i).$$

By (2.5), we get

$$(2.7)\qquad \chi(C) = \sum_{W\in\widehat{G}} e(C_W)\frac{\chi(W)}{\mathrm{rk}(W)}.$$

If $\lambda$ is a complex line, let $\lambda^{-1}$ be the dual line. If $E$ is a finite dimensional complex vector space, set

$$(2.8)\qquad \mathrm{det} E = \Lambda^{\max}(E).$$

Put

$$\mathrm{det} C = \bigotimes_{i=0}^{n} \left(\mathrm{det} C^i\right)^{(-1)^i},$$

$$(2.9)\qquad \mathrm{det} C_W = \bigotimes_{i=0}^{n} \left(\mathrm{det} C_W^i\right)^{(-1)^i}.$$

By (2.5), we obtain

$$(2.10)\qquad \mathrm{det} C = \bigotimes_{W\in\widehat{G}} \mathrm{det} C_W.$$

For $0 \le i \le n$, $C_W^i$ is a vector subspace of $C^i$. Let $b_{C_W^i}$ be the induced symmetric bilinear form on $C_W^i$. let $b_{\mathrm{det} C_W^i}$ be the symmetric bilinear form on $\mathrm{det} C_W^i$ induced by $b_{C_W^i}$, and let $b_{(\mathrm{det} C_W^i)^{-1}}$ be the dual symmetric bilinear form

on $(\det C_W^i)^{-1}$. Also we have symmetric bilinear forms $b_{\det C_W}$ on $\det C_W$ and $b_{\det C}$ on $\det C$.

Put

$$\det(C,G) = \bigoplus_{W\in\widehat{G}} \det C_W. \tag{2.11}$$

**Definition 2.1.** We introduce the formal product

$$b_{\det(C,G)} = \prod_{W\in\widehat{G}} (b_{\det C_W})^{\frac{\chi(W)}{\mathrm{rk}(W)}}. \tag{2.12}$$

For $W \in \widehat{G}$, let $x_W,\ y_W \in \det C_W$, $x_W \neq 0$, $y_W \neq 0$. Set $x = \oplus_{W\in\widehat{G}} x_W$, $y = \oplus_{W\in\widehat{G}} y_W \in \det(C,G)$. Then by definition,

$$b_{\det(C,G)}(x,y) = \prod_{W\in\widehat{G}} (b_{\det C_W}(x_W,y_W))^{\frac{\chi(W)}{\mathrm{rk}(W)}}. \tag{2.13}$$

Tautologically, (2.13) is an identity of characters on $G$. In particular

$$b_{\det(C,G)}(x,y)(1) = \prod_{W\in\widehat{G}} b_{\det C_W}(x_W,y_W). \tag{2.14}$$

In fact (2.14) just says that

$$b_{\det(C,G)}(1) = b_{\det C}. \tag{2.15}$$

Of course, using the orthogonality of the $\chi_W$'s, knowing the formal product $b_{\det(C,G)}$ is equivalent to knowing the symmetric bilinear forms $b_{\det C_W}$.

Clearly

$$H(C_W,\partial_W) = \mathrm{Hom}_G(W, H(C,\partial)) \otimes W,$$

$$H(C,\partial) = \bigoplus_{W\in\widehat{G}} H(C_W,\partial_W). \tag{2.16}$$

For $W \in \widehat{G}$, we define $\det H(C_W,\partial_W)$ as in (2.9). Set

$$\det(H(C,\partial),G) = \bigoplus_{W\in\widehat{G}} \det H(C_W,\partial_W). \tag{2.17}$$

For $W \in \widehat{G}$, there is a canonical isomorphism (cf. [KM] and [BGS, Section 1a)])

$$\det C_W \simeq \det H(C_W,\partial_W). \tag{2.18}$$

From (2.18), we get

$$\det(C,G) \simeq \det(H(C,\partial),G). \tag{2.19}$$

Let $b_{\det H(C_W,\partial_W)}$ be the symmetric bilinear form on $\det H(C_W,\partial_W)$ corresponding to $b_{\det C_W}$ via the canonical isomorphism (2.18).

**Definition 2.2.** we introduce the formal product

$$b_{\det(H(C,\partial),G)} = \prod_{W \in \widehat{G}} \left( b_{\det H(C_W,\partial_W)} \right)^{\frac{\chi(W)}{\mathrm{rk} W}}. \tag{2.20}$$

Tautologically, under the identification (2.19),

$$b_{\det(C,G)} = b_{\det H((C,\partial),G)}. \tag{2.21}$$

By an abuse of notation, we will call the formal product $b_{\det(C,G)}$ a symmetric bilinear form on $\det(C,G)$.

## 2.2 The Thom-Smale complex of a gradient field

Let $M$ be a closed smooth manifold, with $\dim M = n$. For simplicity, we make the assumption that $M$ is oriented.

Let $(F, \nabla^F)$ be a complex flat vector bundle over $M$ carrying the flat connection $\nabla^F$. We make the assumption that $F$ carries a nondegenerate symmetric bilinear form $b^F$.

Let $(F^*, \nabla^{F^*})$ be the dual complex flat vector bundle of $(F, \nabla^F)$ carrying the dual flat connection $\nabla^{F^*}$.

Let $f : M \to \mathbf{R}$ be a Morse function. Let $g^{TM}$ be a Riemannian metric on $TM$ such that the corresponding gradient vector field $-X = -\nabla f \in \Gamma(TM)$ satisfies the Smale transversality conditions (cf. [Sm]), that is, the unstable cells (of $-X$) intersect transversally with the stable cells.

Set

$$B = \{x \in M; X(x) = 0\}. \tag{2.22}$$

For any $x \in B$, let $W^u(x)$ (resp. $W^s(x)$) denote the unstable (resp. stable) cell at $x$, with respect to $-X$. We also choose an orientation $O_x^-$ (resp. $O_x^+$) on $W^u(x)$ (resp. $W^s(x)$).

Let $x$, $y \in B$ satisfy the Morse index relation $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$, then $\Gamma(x, y) = W^u(x) \cap W^s(y)$ consists of a finite number of integral curves $\gamma$ of $-X$. Moreover, for each $\gamma \in \Gamma(x, y)$, by using the orientations chosen above, on can define a number $n_\gamma(x, y) = \pm 1$ as in [BZ1, (1.28)].

If $x \in B$, let $[W^u(x)]$ be the complex line generated by $W^u(x)$. Set

$$C_*(W^u, F^*) = \bigoplus_{x \in B} [W^u(x)] \otimes F_x^*, \tag{2.23}$$

$$C_i(W^u, F^*) = \bigoplus_{x \in B,\ \mathrm{ind}(x) = i} [W^u(x)] \otimes F_x^*. \tag{2.24}$$

If $x \in B$, the flat vector bundle $F^*$ is canonically trivialized on $W^u(x)$. In particular, if $x$, $y \in B$ satisfy $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$, and if $\gamma \in \Gamma(x, y)$, $f^* \in F_x^*$, let $\tau_\gamma(f^*)$ be the parallel transport of $f^* \in F_x^*$ into $F_y^*$ along $\gamma$ with respect to the flat connection $\nabla^{F^*}$.

Clearly, for any $x \in B$, there is only a finite number of $y \in B$, satisfying together that $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$ and $\Gamma(x, y) \neq \emptyset$.

If $x \in B$, $f^* \in F_x^*$, set

$$\partial(W^u(x) \otimes f^*) = \sum_{y \in B,\ \mathrm{ind}(y) = \mathrm{ind}(x)-1} \ \sum_{\gamma \in \Gamma(x,y)} n_\gamma(x, y) W^u(y) \otimes \tau_\gamma(f^*). \tag{2.25}$$

Then $\partial$ maps $C_i(W^u, F^*)$ into $C_{i-1}(W^u, F^*)$. Moreover, one has

$$\partial^2 = 0. \tag{2.26}$$

That is, $(C_*(W^u, F^*), \partial)$ forms a chain complex. We call it the Thom-Smale complex associated to $(M, F, -X)$.

If $x \in B$, let $[W^u(x)]^*$ be the dual line to $W^u(x)$. Let $(C^*(W^u, F), \partial)$ be the complex which is dual to $(C_*(W^u, F^*), \partial)$. For $0 \leq i \leq n$, one has

$$C^i(W^u, F) = \bigoplus_{x \in B,\ \mathrm{ind}(x) = i} [W^u(x)]^* \otimes F_x. \tag{2.27}$$

Let $G$ be a compact group acting on $M$ by smooth diffeomorphisms. we assume that the action of $G$ lifts to $F$ and preserves the flat connection of $F$. Then $G$ acts naturally on $H^*(M, F)$. We assume that $f$ and $g^{TM}$ are $G$-invariant. Then $-X = -\nabla f$ is also $G$-invariant. We assume that it verifies the smale transversality conditions.

Clearly $B$ is $G$-invariant. Also if $x \in B$, $g \in G$,

$$g\left(W^u(x)\right) = \epsilon_g(x) W^u(gx),$$

where $\epsilon_g(x) = +1$ if $g(W^u(x))$ has the same orientation as $W^u(gx)$, $\epsilon_g = -1$ if not. Clearly $g$ acts as a chain homomorphism on $(C_*(W^u, F^*), \partial)$. The corresponding dual action of $g$ on $(C^*(W^u, F), \partial)$ is such that

$$g\left(W^u(x)^*\right) = \epsilon_g(x) W^u(gx)^*.$$

Then $g$ acts as a chain homomorphism on $(C^*(W^u, F), \partial)$. Therefore $g$ acts on $H^*(C^*(W^u, F), \partial)$.

### 2.3 Equivariant Milnor symmetric bilinear torsion

For $x \in B$, let $b^{F_x}$ be a nondegenerate symmetric bilinear form on $F_x$. We assume that the $b^{F_x}$'s are $G$-invariant, i.e. for $g \in G$, $x \in B$

$$g\left(b^{F_x}\right) = b^{F_{g(x)}}. \tag{2.28}$$

The symmetric bilinear forms $b^{F_x}$'s determine a $G$-invariant symmetric bilinear form on $C^*(W^u, F) = \bigoplus_{x \in B} [W^u(x)]^* \otimes F_x$, such that the various $[W^u(x)]^* \otimes F_x$ are mutually orthogonal in $C^*(W^u, F)$, and that if $x \in B$, $f,\ f' \in F_x$,

$$\left\langle W^u(x)^* \otimes f, W^u(x)^* \otimes f' \right\rangle = \left\langle f, f' \right\rangle_{b^{F_x}}. \tag{2.29}$$

We construct the equivariant symmetric bilinear form $b_{\det(C^*(W^u, F), G)}$ on $\det(C^*(W^u, F), G)$ as in Definition 2.1.

**Definition 2.3.** The symmetric bilinear form on the determinant line of the cohomology of the Thom-Smale cochain complex $(C^*(W^u, F), \partial)$, in the sence of Definition 2.2, is called the equivariant Milnor symmetric bilinear torsion and is denoted by $b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}$.

Take $g \in G$. Set

$$M_g = \{x \in M,\ gx = x\}. \tag{2.30}$$

Since $G$ is a compact group, $M_g$ is a smooth compact submanifold of $M$. Let $N$ be the normal bundle to $M_g$ in $M$. By [BZ2, Proposition 1.13], we know that $f|_{M_g}$ is a Morse function on $M_g$, and $X|_{M_g}$ is a smooth section of $TM_g$. For $g \in G$, set

$$B_g = B \cap M_g. \tag{2.31}$$

Then $B_g$ is the set of critical points of $f|_{M_g}$.

**Definition 2.4.** If $x \in B_g$, let $\mathrm{ind}_g(x)$ be the index of $f|_{M_g}$ at $x$.

Let now $b^{F_x}$, $b'^{F_x}$ $(x \in B)$ be two $G$-invariant nondegenerate symmetric bilinear forms on $F_x$. Let $b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}$, $b'^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}$ be the corresponding equivariant Milnor symmetric bilinear torsions. By [BZ2, Theorem 1.15] and [SZ, Proposition 2.5], we have the following theorem.

**Theorem 2.5.** *For $g \in G$, the following identity holds*

$$b'^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}(g) = b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}(g) \prod_{x \in B_g} \exp\left(\mathrm{Tr}_{F_x}\left[g \log\left(\frac{b'^{F_x}}{b^{F_x}}\right)\right]\right)^{(-1)^{\mathrm{ind}_g(x)}}. \tag{2.32}$$

## 2.4 Equivariant Ray-Singer symmetric bilinear torsion

We continue the discussion of the previous subsection. However, we do not use the Morse function and make transversality assumptions.

For any $0 \le i \le n$, denote

$$\Omega^i(M, F) = \Gamma\left(\Lambda^i(T^*M) \otimes F\right), \qquad \Omega^*(M, F) = \bigoplus_{i=0}^{n} \Omega^i(M, F). \tag{2.33}$$

Let $d^F$ denote the natural exterior differential on $\Omega^*(M, F)$ induced from $\nabla^F$ which maps each $\Omega^i(M, F)$, $0 \le i \le n$, into $\Omega^{i+1}(M, F)$.

The group $G$ acts naturally on $\Omega^*(M, F)$. Namely, if $g \in G$, $s \in \Omega^*(M, F)$, set

$$gs(x) = g_* s(g^{-1}x),\ x \in M.$$

Let $g^F$ be a $G$-invariant Hermitian metric on $F$. The $G$-invariant Riemannian metric $g^{TM}$ and $g^F$ determine a natural inner product $\langle\ ,\ \rangle_g$ (that is, a pre-Hilbert space structure) on $\Omega^*(M, F)$ (cf. [BZ1, (2.2)] and [BZ2, (2.3)]).

Let $d^{F*}_g$ be the formal adjoint of $d^F$ with respect to $\langle\ ,\ \rangle_g$ and $D_g = d^F + d^{F*}_g$.

On the other hand $g^{TM}$ and the $G$-invariant symmetric bilinear form $b^F$ determine together a $G$-invariant symmetric bilinear form on $\Omega^*(M,F)$ such that if $u=\alpha f$, $v=\beta g\in\Omega^*(M,F)$ such that $\alpha,\ \beta\in\Omega^*(M)$, $f,\ g\in\Gamma(F)$, then

$$\langle u,v\rangle_b=\int_M(\alpha\wedge*\beta)b^F(f,g), \tag{2.34}$$

where $*$ is the Hodge star operator (cf. [Z]).

Consider the de Rham complex

$$\begin{aligned}(\Omega^*(M,F),d^F):0\to\Omega^0(M,F)\xrightarrow{d^F}\Omega^1(M,F)\to\cdots\\ \xrightarrow{d^F}\Omega^n(M,F)\to 0.\end{aligned} \tag{2.35}$$

Let $d_b^{F*}:\Omega^*(M,F)\to\Omega^*(M,F)$ denote the formal adjoint of $d^F$ with respect to $G$-invariant the symmetric bilinear form in (2.34). That is, for any $u,\ v\in\Omega^*(M,F)$, one has

$$\left\langle d^Fu,v\right\rangle_b=\left\langle u,d_b^{F*}v\right\rangle_b. \tag{2.36}$$

Set

$$D_b=d^F+d_b^{F*},\quad D_b^2=\left(d^F+d_b^{F*}\right)^2=d_b^{F*}d^F+d^Fd_b^{F*}. \tag{2.37}$$

Then the Laplacian $D_b^2$ preserves the $\mathbf{Z}$-grading of $\Omega^*(M,F)$.

As was pointed out in [BH1] and [BH2], $D_b^2$ has the same principal symbol as the usual Hodge Laplacian (constructed using the inner product on $\Omega^*(M,F)$ induced from $(g^{TM},g^F)$) studied for example in [BZ1].

We collect some well-known facts concerning $D_b^2$ as in [BH2, Proposition 4.1], where the reference [S] is indicated.

**Proposition 2.6.** *The following properties hold for the Laplacian $D_b^2$:*

*(i) The spectrum of $D_b^2$ is discrete. For every $\theta>0$ all but finitely many points of the spectrum are contained in the angle $\{z\in\mathbf{C}|-\theta<\arg(z)<\theta\}$;*

*(ii) If $\lambda$ is in the spectrum of $D_b^2$, then the image of the associated spectral projection is finite dimensional and contains smooth forms only. We refer to this image as the (generalized) $\lambda$-eigen space of $D_b^2$ and denote it by $\Omega^*_{\{\lambda\}}(M,F)$. There exists $N_\lambda\in\mathbf{N}$ such that*

$$\left(D_b^2-\lambda\right)^{N_\lambda}\Big|_{\Omega^*_{\{\lambda\}}(M,F)}=0. \tag{2.38}$$

*We have a $D_b^2$-invariant $\langle\ ,\ \rangle_b$-orthogonal decomposition*

$$\Omega^*(M,F)=\Omega^*_{\{\lambda\}}(M,F)\oplus\Omega^*_{\{\lambda\}}(M,F)^\perp. \tag{2.39}$$

*The restriction of $D_b^2-\lambda$ to $\Omega^*_{\{\lambda\}}(M,F)^\perp$ is invertible;*

*(iii) The decomposition (2.39) is invariant under $d^F$ and $d_b^{F*}$;*

*(iv) For $\lambda\neq\mu$, the eigen spaces $\Omega^*_{\{\lambda\}}(M,F)$ and $\Omega^*_{\{\mu\}}(M,F)$ are $\langle\ ,\ \rangle_b$-orthogonal to each other.*

For any $a \geq 0$, set

$$\Omega^*_{[0,a]}(M,F) = \bigoplus_{0\leq|\lambda|\leq a} \Omega^*_{\{\lambda\}}(M,F). \tag{2.40}$$

Let $\Omega^*_{[0,a]}(M,F)^{\perp}$ denote the $\langle\ ,\ \rangle_b$-orthogonal complement to $\Omega^*_{[0,a]}(M,F)$. Obviously, each $\Omega^*_{\{\lambda\}}(M,F)$ is a $G$-invariant subspace.

By [BH2, (29)] and Proposition 2.6, one sees that $(\Omega^*_{[0,a]}(M,F), d^F)$ forms a finite dimensional complex whose cohomology equals to that of $(\Omega^*(M,F), d^F)$. Moreover, the $G$-invariant symmetric bilinear form $\langle\ ,\ \rangle_b$ clearly induces a nondegenerate $G$-invariant symmetric bilinear form on each $\Omega^i_{[0,a]}(M,F)$ with $0 \leq i \leq n$. By Definition 2.2 one then gets a symmetric bilinear torsion $b^{\mathrm{RS}}_{\det(H^*(\Omega^*_{[0,a]}(M,F)),G)}$ on $\det H^*(\Omega^*_{[0,a]}(M,F), d^F) = \det H^*(\Omega^*(M,F), d^F)$.

For any $0 \leq i \leq n$, let $D^2_{b,i}$ be the restriction of $D^2_b$ on $\Omega^i(M,F)$. Then it is shown in [BH2] (cf. [S, Theorem 13.1]) that for any $a \geq 0$, $g \in G$ the following is well-defined,

$$\det{}'\left(D^2_{b,(a,+\infty),i}\right)(g) = \exp\left(-\left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{Tr}\left[g\left(D^2_{b,i}|_{\Omega^*_{[0,a]}(M,F)^{\perp}}\right)^{-s}\right]\right). \tag{2.41}$$

**Definition 2.7.** If $g \in G$, set

(2.42)
$$b^{\mathrm{RS}}_{\det(H^*(M,F),G)}(g) = b^{\mathrm{RS}}_{\det(H^*(\Omega^*_{[0,a]}(M,F)),G)}(g) \prod_{i=0}^{n} \left(\det{}'\left(D^2_{b,(a,+\infty),i}\right)(g)\right)^{(-1)^i i},$$

by [BH2, Proposition 4.7], we know that $b^{\mathrm{RS}}_{\det(H^*(M,F),G)}$ does not depend on the choice of $a \geq 0$, and is called the equivariant Ray-Singer symmetric bilinear torsion on $\det H^*(\Omega^*(M,F), d^F)$.

## 2.5 An anomaly formula for the equivariant Ray-Singer symmetric bilinear torsion

We continue the discussion of the above subsection.

**Definition 2.8.** Let $\theta_g(F, b^F)$ be the 1-form on $M_g$

$$\theta_g(F, b^F) = \mathrm{Tr}\left[g(b^F)^{-1}\nabla^F b^F\right]. \tag{2.43}$$

Clearly $M_g$ is a totally geodesic submanifold of $M$. Let $g^{TM_g}$ be the Riemannian metric induced by $g^{TM}$ on $TM_g$. Let $\nabla^{TM_g}$ be the Levi-Civita connection on $(TM_g, g^{TM_g})$.

Let $e(TM_g, \nabla^{TM_g})$ be the Chern-Weil representative of the rational Euler class of $TM_g$, associated to the metric preserving connection $\nabla^{TM_g}$. Then

$$e(TM_g, \nabla^{TM_g}) = \mathrm{Pf}\left[\frac{R^{TM_g}}{2\pi}\right] \text{ if } \dim M_g \text{ is even,} \tag{2.44}$$

0 if $\dim M_g$ is odd.

Let $g'^{TM}$ be another $G$-invariant metric and let $\nabla'^{TM_g}$ be the corresponding Levi-Civita connection on $TM_g$. Let $\widetilde{e}(TM_g, \nabla^{TM_g}, \nabla'^{TM_g})$ be the Chern-Simons class of $\dim M_g - 1$ forms on $M_g$, such that

$$d\widetilde{e}\left(TM_g, \nabla^{TM_g}, \nabla'^{TM_g}\right) = e\left(TM_g, \nabla'^{TM_g}\right) - e\left(TM_g, \nabla^{TM_g}\right). \tag{2.45}$$

Let $b'^F$ be another $G$-invariant nondegenerate symmetric bilinear form on $F$.

Let $b'^{\mathrm{RS}}_{\det(H^*(M,F),G)}$ denote the equivariant Ray-Singer symmetric bilinear torsion associated to $g'^{TM}$ and $b'^F$.

By [SZ, Remark 6.4] and [BZ2, Theorem 2.7], we have the following extension of the anomaly formula of [SZ, Theorem 2.9].

**Theorem 2.9.** *If $b^F$, $b'^F$ lie in the same homotopy class of nondegenerate symmetric bilinear forms on $F$, then for $g \in G$ the following identity holds,*

$$\begin{aligned}\left(\frac{b'^{\mathrm{RS}}_{\det(H^*(M,F),G)}}{b^{\mathrm{RS}}_{\det(H^*(M,F),G)}}\right)(g) = \exp\left(\int_{M_g} \mathrm{Tr}\left[g \log\left(\frac{b'^F}{b^F}\right)\right] e\left(TM_g, \nabla^{TM_g}\right)\right) \\ \cdot \exp\left(-\int_{M_g} \theta_g(F, b'^F)\widetilde{e}\left(TM_g, \nabla^{TM_g}, \nabla'^{TM_g}\right)\right).\end{aligned} \tag{2.46}$$

*Proof.* Let $b^F_l$ is a smooth one-parameter family of fiber wise non-degenerate symmetric bilinear forms on $F$ and $(g_l^{TM}, g_l^F)$ be a smooth family of metrics on $TM, F$.

By [SZ, (6.4)], we have

$$\begin{aligned}e^{-tD_b^2} = e^{-tD_g^2} + \sum_{k=1}^{n}(-1)^k t^k \int_{\Delta_k} e^{-t_1 tD_g^2} B_{b,g} e^{-t_2 tD_g^2} \cdots B_{b,g} e^{-t_{k+1} tD_g^2} dt_1 \cdots dt_k \\ + (-1)^{n+1} t^{n+1} \int_{\Delta_{n+1}} e^{-t_1 tD_g^2} B_{b,g} e^{-t_2 tD_g^2} \cdots B_{b,g} e^{-t_{n+2} tD_b^2} dt_1 \cdots dt_{n+1},\end{aligned} \tag{2.47}$$

where $\Delta_k$, $1 \le k \le n+1$, is the $k$-simplex defined by $t_1 + \cdots + t_{k+1} = 1$, $t_1 \ge 0$, $\cdots$, $t_{k+1} \ge 0$ and $B_{b,g}$ is defined in [SZ, (6.3)].

Proceeding as in [BZ1, Section 4], we first calculate the asymptotics as $t \to 0$ of $\mathrm{Tr}_s[g(b_l^F)^{-1}\frac{\partial b_l^F}{\partial l}\exp(-tD^2_{b_l})]$. Here the metric $g^{TM}$ will be fixed.

By the same proof in [SZ, proposition 6.1], we have that as $t \to 0^+$,

$$t^{n+1}\int_{\Delta_{n+1}} \mathrm{Tr}_s\left[g(b_l^F)^{-1}\frac{\partial b_l^F}{\partial l} e^{-t_1 tD_g^2} B_{b,g} e^{-t_2 tD_g^2} \cdots B_{b,g} e^{-t_{n+2} tD_b^2}\right] dt_1 \cdots dt_{n+1} \to 0. \tag{2.48}$$

Also, by [SZ, (6.22)], we have that for $1 \le k \le n$, $(t_1, \cdots, t_{k+1}) \in \Delta_k$,

$$\lim_{t \to 0^+} t^k \mathrm{Tr}_s\left[g(b_l^F)^{-1}\frac{\partial b_l^F}{\partial l} e^{-t_1 tD_g^2} B_{b,g} e^{-t_2 tD_g^2} \cdots B_{b,g} e^{-t_{k+1} tD_g^2}\right] = 0. \tag{2.49}$$

So that by (2.47)-(2.49) we have that

(2.50)
$$\lim_{t\to 0}\mathrm{Tr}_s\left[g(b_l^F)^{-1}\frac{\partial b_l^F}{\partial l}\exp\left(-tD_{b_l}^2\right)\right]=\lim_{t\to 0}\mathrm{Tr}_s\left[g(b_l^F)^{-1}\frac{\partial b_l^F}{\partial l}\exp\left(-tD_{g_l}^2\right)\right].$$

Now we assume that the nondegenerate symmetric bilinear form on $F$ is fixed, and the metric $g_l^{TM}$ on $TM$ depends on $l$.

Let $*_l$ be the Hodge star operator associated to $g_l^{TM}$.

By [BZ1, (4.70), (4.74)], analogues of [SZ, (6.5), (6.24), (6.26), (6.27)] replacing $N$ by $*_l^{-1}\frac{\partial *_l^{TM}}{\partial l}$ and [BGV, Chapter 6], we have that

(2.51)
$$\begin{aligned}
&\lim_{t\to 0}\mathrm{Tr}_s\left[g\,*_l^{-1}\frac{\partial *_l^{TM}}{\partial l}\exp\left(-tD_{b_l}^2\right)\right]=\lim_{t\to 0}\mathrm{Tr}_s\left[g\,*_l^{-1}\frac{\partial *_l^{TM}}{\partial l}\exp\left(-tD_{g_l}^2\right)\right]\\
&\quad-\frac{1}{2}\int_{M_g}\int^B\mathrm{Tr}\left[g\left(\sum_{i,j=1}^n e_i\wedge\widehat{e_j}\left(\nabla_{e_i}^u\omega^F\left(e_j\right)\right)+\frac{1}{2}\left[\omega^F,\widehat{\omega_g^F}-\widehat{\omega^F}\right]\right)\right]\\
&\quad\cdot\left(-\sum_{1\le i,j\le n}\frac{1}{2}\left\langle\left(g_l^{TM}\right)^{-1}\frac{\partial g_l^{TM}}{\partial l}e_i,e_j\right\rangle_{g_l^{TM}}e_i\wedge\widehat{e_j}\right)\exp\left(-\frac{\dot{R}_l^{TM_g}}{2}\right)\\
&\quad=\lim_{t\to 0}\mathrm{Tr}_s\left[g\,*_l^{-1}\frac{\partial *_l^{TM}}{\partial l}\exp\left(-tD_{g_l}^2\right)\right]-\frac{1}{2}\int_{M_g}\int^B\nabla_l^{TM}\varphi\mathrm{Tr}\left[g\omega^F\right]\\
&\quad\cdot\left(-\sum_{1\le i,j\le n}\frac{1}{2}\left\langle\left(g_l^{TM}\right)^{-1}\frac{\partial g_l^{TM}}{\partial l}e_i,e_j\right\rangle_{g_l^{TM}}e_i\wedge\widehat{e_j}\right)\exp\left(-\frac{\dot{R}_l^{TM_g}}{2}\right)\\
&\quad=\int_{M_g}\left\{\int^B\nabla_l^{TM}\left(\frac{1}{4}\sum_{1\le i,j\le n}\left\langle\left(g_l^{TM}\right)^{-1}\frac{\partial g_l^{TM}}{\partial l}e_i,e_j\right\rangle_{g_l^{TM}}e_i\wedge\widehat{e_j}\right)\right.\\
&\qquad\qquad\left.\cdot\exp\left(-\frac{\dot{R}_l^{TM_g}}{2}\right)\wedge\varphi\theta_g\left(F,b^F\right)\right\}.
\end{aligned}$$

From (2.50), (2.51) and the calculations in [BZ1, Section 4], we get (2.46). The proof of Theorem 2.9 is completed. Q.E.D.

# 3 A formula relating equivariant Milnor and equivariant Ray-Singer symmetric bilinear torsions

In this section, we state the main result of this paper, which is an explicit comparison result between the equivariant Milnor symmetric bilinear torsion and equivariant Ray-Singer symmetric bilinear torsion.

We assume that we are in the same situation as in Sections 2.2-2.4. By a simple argument of Helffer-Sjöstrand [HS, Proposition 5.1] (cf. [BZ1, Section 7b)]), we may and we well assume that $g^{TM}$ there satisfies the following property without altering the Thom-Smale cochain complex $(C^*(W^u,F),\partial)$,

(*): For any $x \in B$, there is a system of coordinates $y = (y^1, \cdots, y^n)$ centered at $x$ such that near $x$,

$$g^{TM} = \sum_{i=1}^{n} \left|dy^i\right|^2, \quad f(y) = f(x) - \frac{1}{2} \sum_{i=1}^{\operatorname{ind}(x)} \left|y^i\right|^2 + \frac{1}{2} \sum_{i=\operatorname{ind}(x)+1}^{n} \left|y^i\right|^2. \tag{3.1}$$

By a result of Laudenbach [L], $\{W^u(x) : x \in B\}$ form a CW decomposition of $M$.

For any $x \in B$, $F$ is canonically trivialized over each cell $W^u(x)$.

Let $P_\infty$ be the de Rham map defined by

$$\alpha \in \Omega^*(M, F) \to P_\infty \alpha = \sum_{x \in B} W^u(x)^* \int_{W^u(x)} \alpha \in C^*(W^u, F). \tag{3.2}$$

By the Stokes theorem, one has

$$\partial P_\infty = P_\infty d^F. \tag{3.3}$$

Moreover, it is shown in [L] that $P_\infty$ is a $\mathbf{Z}$-graded quasi-isomorphism, inducing a canonical isomorphism

$$P_\infty^H : H^*\left(\Omega^*(M, F), d^F\right) \to H^*\left(C^*\left(W^u, F\right), \partial\right), \tag{3.4}$$

which in turn induces a natural isomorphism between the determinant lines,

$$P_\infty^{\det H} : \det H^*\left(\Omega^*\left(M, F\right), d^F\right) \to \det H^*\left(C^*\left(W^u, F\right), \partial\right). \tag{3.5}$$

Also by [BZ2, Theorem 1.11], we know that $P_\infty$ commutes with $G$, and $P_\infty^H$ is the canonical identification of the corresponding cohomology groups as $G$-spaces.

Now let $h^{TM}$ be an arbitrary smooth metric on $TM$.

By Definition 2.7, one has an associated equivariant Ray-Singer symmetric bilinear torsion $b^{\mathrm{RS}}_{\det(H^*(M,F),G)}$ on $\det H^*(\Omega^*(M, F), d^F)$. From (3.5), one gets a well-defined equivariant symmetric bilinear form

$$P_\infty^{\det H}\left(b^{\mathrm{RS}}_{\det(H^*(M,F),G)}\right) \tag{3.6}$$

on $\det H^*(C^*(W^u, F), \partial)$.

On the other hand, by Definition 2.3, one has a well-defined equivariant Milnor symmetric bilinear torsion $b^{\mathcal{M},-X}_{\det(H^*(M,F),G)}$ on $\det H^*(C^*(W^u, F), \partial)$, where $X = \nabla f$ is the gradient vector field of $f$ associated to $g^{TM}$.

Let $M_g = \cup_{j=1}^m M_{g,j}$ be the decomposition of $M_g$ into its connected components. Clearly $\mathrm{Tr}_F[g]$ is constant on each $M_{g,j}$.

Let $N$ be the normal bundle to $M_g$ in $M$. We identify $N$ to the orthogonal bundle to $TM_g$ in $TM|_{M_g}$.

Take $x \in B_g$. Then $g$ acts on $T_xM$ as a linear isometry. Also

$$TM_g = \{Y \in TM|_{M_g},\ gY = Y\}.$$

Moreover $g$ acts on $N$. Let $e^{\pm i\beta_1}, \cdots, e^{\pm i\beta_q}(0 < \beta_j \leq \pi)$ be the locally constant distinct eigenvalues of $g|_N$. Then $N$ splits orthogonally as

$$N = \bigoplus_{j=1}^{q} N^{\beta_j}.$$

For $1 \leq j \leq q$, $g$ acts on $N^{\beta_j}$ as an isometry, with eigenvalues $e^{\pm i\beta_j}$. In particular, if $e^{\pm i\beta_j} \neq -1$, $N^{\beta_j}$ is even dimensional.

Take $x \in B_g$. Since $f$ is $g$-invariant, $d^2 f(x)$ is also $g$-invariant. Therefore the decomposition

$$T_x M = T_x M_g \oplus \bigoplus_{j=1}^{q} N^{\beta_j}$$

is orthogonal with respect to $d^2 f(x)$. On $T_x M_g$, the index of $d^2 f(x)|_{T_x M_g \times T_x M_g}$ was already denoted $\operatorname{ind}_g(x)$. Let $n_+(\beta_j)(x)$ (resp. $n_-(\beta_j)(x)$) be the number of positive (resp. negative) eigenvalues of $d^2 f(x)|_{N^{\beta_j}}$. Then if $e^{\pm i\beta_j} \neq -1, n_\pm(\beta_j)(x)$ is even.

Let $\psi(TM_g, \nabla^{TM_g})$ be the Mathai-Quillen current ([MQ]) over $TM_g$, associated to $h^{TM}$, defined in [BZ1, Definition 3.6]. As indicated in [BZ1, Remark 3.8], the pull-back current $X^*\psi(TM_g, \nabla^{TM_g})$ is well-defined over $M_g$.

The main result of this paper, which generalizes [SZ, Theorem 3.1] to the equivariant case.

**Theorem 3.1.** *For $g \in G$, the following identity in $\mathbf{C}$ holds,*

(3.7)

$$\frac{P_\infty^{\det H}\left(b^{\mathrm{RS}}_{\det(H^*(M,F),G)}\right)}{b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}}(g) = \exp\left(-\int_{M_g} \theta_g(F,b^F) X^*\psi\left(TM_g, \nabla^{TM_g}\right)\right)$$
$$\cdot \exp\left(-\frac{1}{4}\sum_{x\in B_g}(-1)^{\operatorname{ind}_g(x)}\sum_j (n_+(\beta_j)(x) - n_-(\beta_j)(x))\right.$$
$$\left.\cdot\left(\frac{\Gamma'}{\Gamma}\left(\frac{\beta_j}{2\pi}\right) + \frac{\Gamma'}{\Gamma}\left(1 - \frac{\beta_j}{2\pi}\right) - 2\Gamma'(1)\right)\cdot \operatorname{Tr}\left[g|_{F_x}\right]\right).$$

*Remark* 3.2. By proceeding similarly as in [BZ2, Section 5b)], in order to prove (3.7), we may well assume that $h^{TM} = g^{TM}$. Moreover, we may assume that $b^F$, as well as the Hermitian metric $h^F$ on $F$, are flat on an open neighborhood of the zero set $B$ of $X$. From now on, we will make these assumptions.

## 4 A proof of Theorem 3.1

We assume that the assumptions in Remark 3.2 hold.

For any $T \in \mathbf{R}$, let $b^F_T$ be the deformed symmetric bilinear form on $F$ defined by

$$b^F_T(u,v) = e^{-2Tf} b^F(u,v). \tag{4.1}$$

Let $d^{F*}_{b_T}$ be the associated formal adjoint in the sense of (2.36). Set

$$D_{b_T} = d^F + d^{F*}_{b_T}, \quad D^2_{b_T} = \left(d^F + d^{F*}_{b_T}\right)^2 = d^{F*}_{b_T} d^F + d^F d^{F*}_{b_T}. \tag{4.2}$$

Let $\Omega^*_{[0,1],T}(M,F)$ be defined as in (2.40) with respect to $D^2_{b_T}$, and let $\Omega^*_{[0,1],T}(M,F)^\perp$ be the corresponding $\langle\ ,\ \rangle_{b_T}$-orthogonal complement.

Let $P^{[0,1]}_T$ be the orthogonal projection from $\Omega^*(M,F)$ to $\Omega^*_{[0,1],T}(M,F)$ with respect to the inner product determined by $g^{TM}$ and $g^F_T = e^{-2Tf} g^F$. Set $P^{(1,+\infty)}_T = \mathrm{Id} - P^{[0,1]}_T$.

Following [BZ2, (5.9)-(5.10)], we introduce the notations

$$\chi_g(F) = \sum_j \mathrm{Tr}_{F|_{M_{g,j}}}[g] \sum_{x \in B_g \cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)}, \tag{4.3}$$

$$\widetilde{\chi}'_g(F) = \sum_j \mathrm{Tr}_{F|_{M_{g,j}}}[g] \sum_{x \in B_g \cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} \mathrm{ind}(x),$$

$$\mathrm{Tr}^{B_g}_s[f] = \sum_j \mathrm{Tr}_{F|_{M_{g,j}}}[g] \sum_{x \in B_g \cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} f(x).$$

Let $N$ be the number operator on $\Omega^*(M,F)$ acting on $\Omega^i(M,F)$ by multiplication by $i$.

By the technique developed in [SZ] and the corresponding results in [BZ2], we easily get the following intermediate results. The sketch of the proofs will be outlined in Section 5.

**Theorem 4.1.** (Compare with [BZ2, Theorem 5.5] and [SZ, Theorem 3.3]) *Let $P^{[0,1]}_T$ be the restriction of $P_\infty$ on $\Omega^*_{[0,1],T}(M,F)$, let $P^{[0,1],\det H}_T$ be the induced isomorphism on cohomology, then the following identity holds,*

$$\lim_{T \to +\infty} \frac{P^{[0,1],\det H}_T \left( b^{\mathrm{RS}}_{\det(H^*(\Omega^*_{[0,1],T}(M,F)),G)} \right)}{b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}}(g) \left(\frac{T}{\pi}\right)^{\frac{n}{2}\chi_g(F) - \widetilde{\chi}'_g(F)} \exp\left(2\mathrm{Tr}^{B_g}_s[f] T\right) = 1. \tag{4.4}$$

**Theorem 4.2.** (Compare with [BZ2, Theorem 5.7] and [SZ, Theorem 3.4]) *For any $t > 0$,*

$$\lim_{T \to +\infty} \mathrm{Tr}_s \left[ gN \exp\left(-tD^2_{b_T}\right) P^{(1,+\infty)}_T \right] = 0. \tag{4.5}$$

*Moreover, for any $d > 0$ there exist $c > 0$, $C > 0$ and $T_0 \geq 1$ such that for any $t \geq d$ and $T \geq T_0$,*

$$\left| \mathrm{Tr}_s \left[ gN \exp\left(-tD^2_{b_T}\right) P^{(1,+\infty)}_T \right] \right| \leq c \exp(-Ct). \tag{4.6}$$

**Theorem 4.3.** (Compare with [BZ2, Theorem 5.8] and [SZ, Theorem 3.5]) *For $T \geq 0$ large enough, then*

$$\lim_{T\to+\infty} \operatorname{Tr}_s\left[gNP_T^{[0,1]}\right] = \widetilde{\chi}_g'(F). \tag{4.7}$$

*Also,*

$$\lim_{T\to+\infty} \operatorname{Tr}\left[D_{b_T}^2 P_T^{[0,1]}\right] = 0. \tag{4.8}$$

For the next results, we will make use the same notation for Clifford multiplications and Berezin integrals as in [BZ1, Section 4].

**Theorem 4.4.** (Compare with [BZ2, Theorem 5.9] and [SZ, Theorem 3.6]) *As $t \to 0$, the following identity holds,*

$$\begin{aligned} \operatorname{Tr}_s\left[gN\exp\left(-tD_{b_T}^2\right)\right] &= \frac{n}{2}\chi_g(F) + O(t) \quad if\ n \text{ is even}, \\ &= \int_{M_g} \operatorname{Tr}_F[g] \int^B L\exp\left(-\frac{\dot{R}^{TM_g}}{2}\right)\frac{1}{\sqrt{t}} + O\left(\sqrt{t}\right) \quad if\ n \text{ is odd}. \end{aligned} \tag{4.9}$$

**Theorem 4.5.** (Compare with [BZ2, Theorem A.1] and [SZ, Theorem 3.7]) *There exist $0 < \alpha \leq 1$, $C > 0$ such that for any $0 < t \leq \alpha$, $0 \leq T \leq \frac{1}{t}$, then*

$$\begin{aligned} \Bigg| \operatorname{Tr}_s\left[gN\exp\left(-\left(tD_b + T\widehat{c}(\nabla f)\right)^2\right)\right] - \frac{1}{t}\int_{M_g} \operatorname{Tr}_F[g]\int^B L\exp\left(-B_{T^2}\right) \\ -\frac{T}{2}\int_{M_g}\theta_g\left(F, b^F\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right) - \frac{n}{2}\chi_g(F)\Bigg| \leq Ct. \end{aligned} \tag{4.10}$$

**Theorem 4.6.** (Compare with [BZ2, Theorem A.2] and [SZ, Theorem 3.8]) *For any $T > 0$, the following identity holds,*

$$\begin{aligned} \lim_{t\to 0} \operatorname{Tr}_s\left[gN\exp\left(-\left(tD_b + \frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] = \sum_j \operatorname{Tr}\left[g|_{F|_{M_{g,j}}}\right] \\ \cdot\left(\frac{1}{1-e^{-2T}}\left(\left(1+e^{-2T}\right)\sum_{x\in B\cap M_{g,j}}(-1)^{\operatorname{ind}_g(x)}\operatorname{ind}_g(x) - \dim M_{g,j}e^{-2T}\chi(M_{g,j})\right)\right) \\ + \sum_j \operatorname{Tr}\left[g|_{F|_{M_{g,j}}}\right]\sum_k \frac{\sinh(2T)}{\cosh(2T)-\cos(\beta_k)}\sum_{x\in B\cap M_{g,j}}(-1)^{\operatorname{ind}_g(x)}n_-(\beta_k)(x) \\ - \sum_j \operatorname{Tr}\left[g|_{F|_{M_{g,j}}}\right]\sum_k \frac{1}{2}\left(\frac{\sinh(2T)}{\cosh(2T)-\cos(\beta_k)} - 1\right)\dim N^{\beta_k}\chi(M_{g,j}). \end{aligned} \tag{4.11}$$

**Theorem 4.7.** (Compare with [BZ2, Theorem A.3] and [SZ, Theorem 3.9]) *There exist $\alpha \in (0,1]$, $c > 0$, $C > 0$ such that for any $t \in (0,\alpha]$, $T \geq 1$, then*

$$\left|\operatorname{Tr}_s\left[gN\exp\left(-\left(tD_b + \frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \widetilde{\chi}_g'(F)\right| \leq c\exp(-CT). \tag{4.12}$$

Clearly, we may and we will assume that the number $\alpha > 0$ in Theorems 4.5 and 4.7 have been chosen to be the same.

Next, we use above theorems to give a proof of Theorem 3.1. Since the process is similar to it in [SZ], so we refer to it for more details.

First of all, by the anomaly formula (2.46), for any $T \geq 0$, $g \in G$, one has

$$\begin{aligned}(4.13)\quad & \frac{P_T^{[0,1],\det H}\left(b^{\mathrm{RS}}_{\det H^*(\Omega^*_{[0,1],T}(M,F),G)}\right)}{b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}}(g) \\ &\quad \cdot \prod_{i=0}^{n}\left(\det\left(D^2_{b_T}|_{\Omega^*_{[0,1],T}(M,F)^\perp\cap\Omega^i(M,F)}\right)(g)\right)^{(-1)^i i} \\ &= \frac{P_\infty^{\det H}\left(b^{\mathrm{RS}}_{\det(H^*(M,F),G)}\right)}{b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}}(g)\exp\left(-2T\int_{M_g}\mathrm{Tr}_F[g]fe\left(TM_g,\nabla^{TM_g}\right)\right).\end{aligned}$$

From now on, we will write $a \simeq b$ for $a,\ b \in \mathbf{C}$ if $e^a = e^b$. Thus, we can rewrite (4.13) as

$$\begin{aligned}(4.14)\quad & \log\left(\frac{P_\infty^{\det H}\left(b^{\mathrm{RS}}_{\det(H^*(M,F),G)}\right)}{b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}}(g)\right) \simeq \log\left(\frac{P_T^{[0,1],\det H}\left(b^{\mathrm{RS}}_{\det H^*(\Omega^*_{[0,1],T}(M,F),G)}\right)}{b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}}(g)\right) \\ &\quad + \sum_{i=0}^{n}(-1)^i i\log\left(\det\left(D^2_{b_T}|_{\Omega^*_{[0,1],T}(M,F)^\perp\cap\Omega^i(M,F)}\right)(g)\right) \\ &\quad + 2T\int_{M_g}\mathrm{Tr}_F[g]fe\left(TM_g,\nabla^{TM_g}\right).\end{aligned}$$

Let $T_0 > 0$ be as in Theorem 4.2. For any $T \geq T_0$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) \geq n+1$, set

$$(4.15)\qquad \theta_{g,T}(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\mathrm{Tr}_s\left[gN\exp\left(-tD^2_{b_T}\right)P_T^{(1,+\infty)}\right]dt.$$

By (4.6), $\theta_{g,T}(s)$ is well defined and can be extended to a meromorphic function which is holomorphic at $s=0$. Moreover,

$$(4.16)\quad \sum_{i=0}^{n}(-1)^i i\log\left(\det\left(D^2_{b_T}|_{\Omega^*_{[0,1],T}(M,F)^\perp\cap\Omega^i(M,F)}\right)(g)\right) \simeq -\left.\frac{\partial\theta_{g,T}(s)}{\partial s}\right|_{s=0}.$$

Let $d = \alpha^2$ with $\alpha$ being as in Theorem 4.7. From (4.15) and Theorems

4.2-4.4, one finds that

$$
\lim_{T\to+\infty} \left.\frac{\partial\theta_{g,T}(s)}{\partial s}\right|_{s=0} = \lim_{T\to+\infty}\int_0^d \left(\operatorname{Tr}_s\left[gN\exp\left(-tD_{b_T}^2\right)\right] - \frac{a_{-1}}{\sqrt{t}} - \frac{n}{2}\chi_g(F)\right)\frac{dt}{t}
- \frac{2a_{-1}}{\sqrt{d}} - \left(\Gamma'(1)-\log d\right)\left(\frac{n}{2}\chi_g(F) - \widetilde{\chi}'_g(F)\right). \tag{4.17}
$$

To study the first term in the right hand side of (4.17), we observe first that for any $T\geq 0$, one has

$$
e^{-Tf}D_{b_T}^2 e^{Tf} = \left(D_b + T\widehat{c}(\nabla f)\right)^2. \tag{4.18}
$$

Thus, one has

$$
\operatorname{Tr}_s\left[N\exp\left(-tD_{b_T}^2\right)\right] = \operatorname{Tr}_s\left[N\exp\left(-t\left(D_b+T\widehat{c}(\nabla f)\right)^2\right)\right]. \tag{4.19}
$$

By (4.19), one writes

$$
\begin{aligned}
&\int_0^d \left(\operatorname{Tr}_s\left[gN\exp\left(-tD_{b_T}^2\right)\right] - \frac{a_{-1}}{\sqrt{t}} - \frac{n}{2}\chi_g(F)\right)\frac{dt}{t} \\
&= 2\int_1^{\sqrt{dT}}\left(\operatorname{Tr}_s\left[gN\exp\left(-\left(\frac{t}{\sqrt{T}}D_b + t\sqrt{T}\widehat{c}(\nabla f)\right)^2\right)\right] - \frac{\sqrt{T}}{t}a_{-1} - \frac{n}{2}\chi_g(F)\right)\frac{dt}{t} \\
&\quad + 2\int_0^{\frac{1}{\sqrt{T}}}\left(\operatorname{Tr}_s\left[gN\exp\left(-\left(tD_b + tT\widehat{c}(\nabla f)\right)^2\right)\right] - \frac{a_{-1}}{t} - \frac{n}{2}\chi_g(F)\right)\frac{dt}{t}.
\end{aligned} \tag{4.20}
$$

In view of Theorem 4.5, we write

$$
\begin{aligned}
&\int_0^{\frac{1}{\sqrt{T}}}\left(\operatorname{Tr}_s\left[gN\exp\left(-\left(tD_b + tT\widehat{c}(\nabla f)\right)^2\right)\right] - \frac{a_{-1}}{t} - \frac{n}{2}\chi_g(F)\right)\frac{dt}{t} \\
&= \int_0^{\frac{1}{\sqrt{T}}}\Bigg(\operatorname{Tr}_s\left[gN\exp\left(-\left(tD_b + tT\widehat{c}(\nabla f)\right)^2\right)\right] - \frac{1}{t}\int_{M_g}\operatorname{Tr}_F[g]\int^B L\exp\left(-B_{(tT)^2}\right) \\
&\qquad - \frac{tT}{2}\int_{M_g}\theta_g\left(F,b^F\right)\int^B \widehat{df}\exp\left(-B_{(tT)^2}\right) - \frac{n}{2}\chi_g(F)\Bigg)\frac{dt}{t} \\
&\qquad + \int_0^{\frac{1}{\sqrt{T}}}\left(\frac{1}{t}\int_{M_g}\operatorname{Tr}_F[g]\int^B L\exp\left(-B_{(tT)^2}\right) - \frac{a_{-1}}{t}\right)\frac{dt}{t} \\
&\qquad + \int_0^{\frac{1}{\sqrt{T}}}\frac{tT}{2}\int_{M_g}\theta_g\left(F,b^F\right)\int^B \widehat{df}\exp\left(-B_{(tT)^2}\right)\frac{dt}{t}.
\end{aligned} \tag{4.21}
$$

By [BZ1, Definitions 3.6, 3.12 and Theorem 3.18], one has, as $T\to+\infty$,

$$
\int_0^{\frac{1}{\sqrt{T}}}\frac{tT}{2}\int_{M_g}\theta_g\left(F,b^F\right)\int^B \widehat{df}\exp\left(-B_{(tT)^2}\right)\frac{dt}{t} \to
\frac{1}{2}\int_{M_g}\theta_g\left(F,b^F\right)(\nabla f)^*\psi\left(TM_g,\nabla^{TM_g}\right). \tag{4.22}
$$

From [BZ1, (3.54)], [SZ, (3.35)] and integration by parts, we have

$$\int_0^{\frac{1}{\sqrt{T}}} \left( \frac{1}{t} \int_{M_g} \mathrm{Tr}_F[g] \int^B L \exp\left(-B_{(tT)^2}\right) - \frac{a_{-1}}{t} \right) \frac{dt}{t} \tag{4.23}$$
$$= -\sqrt{T} \int_{M_g} \mathrm{Tr}_F[g] \int^B L \exp\left(-B_T\right) + \sqrt{T} a_{-1} - T \int_{M_g} \mathrm{Tr}_F[g] f \int^B \exp\left(-B_T\right)$$
$$+ T \int_{M_g} \mathrm{Tr}_F[g] f \int^B \exp\left(-B_0\right).$$

From Theorems 4.5, 4.6, [BZ1, Theorem 3.20], [BZ1, (7.72) and (7.73)] and the

dominate convergence, one finds that as $T \to +\infty$,

(4.24)

$$\begin{aligned}
&\int_0^{\frac{1}{\sqrt{T}}} \Bigg( \mathrm{Tr}_s \left[ gN \exp\left( - (tD_b + tT\widehat{c}(\nabla f))^2 \right) \right] - \frac{1}{t} \int_{M_g} \mathrm{Tr}_F[g] \int^B L \exp\left( -B_{(tT)^2} \right) \\
&\qquad - \frac{tT}{2} \int_{M_g} \theta_g\left(F, b^F\right) \int^B \widehat{df} \exp\left(-B_{(tT)^2}\right) - \frac{n}{2}\chi_g(F) \Bigg) \frac{dt}{t} \\
&= \int_0^1 \Bigg( \mathrm{Tr}_s \left[ gN \exp\left( - \left( \frac{t}{\sqrt{T}} D_b + t\sqrt{T}\widehat{c}(\nabla f) \right)^2 \right) \right] \\
&\qquad - \frac{\sqrt{T}}{t} \int_{M_g} \mathrm{Tr}_F[g] \int^B L \exp\left( -B_{(t\sqrt{T})^2} \right) \\
&\qquad - \frac{t\sqrt{T}}{2} \int_{M_g} \theta_g\left(F, b^F\right) \int^B \widehat{df} \exp\left(-B_{(t\sqrt{T})^2}\right) - \frac{n}{2}\chi_g(F) \Bigg) \frac{dt}{t} \\
&\to \int_0^1 \Bigg\{ \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \\
&\cdot \left( \frac{1}{1-e^{-2t^2}} \left( \left(1+e^{-2t^2}\right) \sum_{x \in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} \mathrm{ind}_g(x) - \mathrm{dim} M_{g,j} e^{-2t^2} \chi(M_{g,j}) \right) \right) \\
&\quad + \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \sum_k \frac{\sinh(2t^2)}{\cosh(2t^2) - \cos(\beta_k)} \sum_{x \in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} n_-(\beta_k)(x) \\
&\quad - \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \sum_k \frac{1}{2} \left( \frac{\sinh(2t^2)}{\cosh(2t^2) - \cos(\beta_k)} - 1 \right) \mathrm{dim} N^{\beta_k} \chi(M_{g,j}) \\
&+ \frac{1}{2t^2} \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \sum_{x \in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} \left( \mathrm{dim} M_{g,j} - 2\mathrm{ind}_g(x) \right) - \frac{n}{2}\chi_g(F) \Bigg\} \frac{dt}{t} \\
&= \frac{1}{2} \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \left\{ \sum_{x \in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} \mathrm{ind}_g(x) - \frac{1}{2} \sum_j \chi(M_{g,j}) \mathrm{dim} M_{g,j} \right\} \\
&\qquad \cdot \int_0^1 \left( \frac{1+e^{-2t}}{1-e^{-2t}} - \frac{1}{t} \right) \frac{dt}{t} \\
&- \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \sum_k \left( \frac{1}{4} \mathrm{dim} N^{\beta_k} \chi(M_{g,j}) - \frac{1}{2} \sum_{x \in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} n_-(\beta_k)(x) \right) \\
&\qquad \cdot \int_0^1 \left( \frac{\sinh(2t)}{\cosh(2t) - \cos(\beta_k)} \right) \frac{dt}{t}.
\end{aligned}$$

On the other hand, by Theorems 4.6, 4.7 and the dominate convergence, we

have that as $T \to +\infty$,

(4.25)
$$
\begin{aligned}
&\int_1^{\sqrt{Td}} \left( \mathrm{Tr}_s \left[ gN \exp\left( -\left( \frac{t}{\sqrt{T}} D_b + t\sqrt{T}\widehat{c}(\nabla f) \right)^2 \right) \right] - \frac{\sqrt{T}}{t} a_{-1} - \frac{n}{2}\chi_g(F) \right) \frac{dt}{t} \\
&= \int_1^{\sqrt{Td}} \left( \mathrm{Tr}_s \left[ gN \exp\left( -\left( \frac{t}{\sqrt{T}} D_b + t\sqrt{T}\widehat{c}(\nabla f) \right)^2 \right) \right] - \widetilde{\chi}_g'(F) \right) \frac{dt}{t} \\
&\quad + \frac{1}{2}\widetilde{\chi}_g'(F)\log(Td) + a_{-1}\sqrt{T}\left( \frac{1}{\sqrt{Td}} - 1 \right) - \frac{n}{4}\chi_g(F)\log(Td) \\
&= \int_1^{+\infty} \Bigg\{ \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \\
&\cdot \left( \frac{1}{1-e^{-2t^2}} \left( \left(1+e^{-2t^2}\right) \sum_{x\in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} \mathrm{ind}_g(x) - \dim M_{g,j} e^{-2t^2} \chi(M_{g,j}) \right) \right) \\
&\quad + \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \sum_k \frac{\sinh(2t^2)}{\cosh(2t^2)-\cos(\beta_k)} \sum_{x\in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} n_-(\beta_k)(x) \\
&- \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \sum_k \frac{1}{2} \left( \frac{\sinh(2t^2)}{\cosh(2t^2)-\cos(\beta_k)} - 1 \right) \dim N^{\beta_k} \chi(M_{g,j}) - \widetilde{\chi}_g'(F) \Bigg\} \frac{dt}{t} \\
&\quad + \frac{1}{2}\widetilde{\chi}_g'(F)\log(Td) + a_{-1}\sqrt{T}\left( \frac{1}{\sqrt{Td}} - 1 \right) - \frac{n}{4}\chi_g(F)\log(Td) + o(1) \\
&= \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \left\{ \sum_{x\in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} \mathrm{ind}_g(x) - \frac{1}{2}\sum_j \chi(M_{g,j}) \dim M_{g,j} \right\} \\
&\quad \cdot \int_1^{+\infty} \frac{e^{-2t}}{1-e^{-2t}} \frac{dt}{t} \\
&- \sum_j \mathrm{Tr}\left[ g|_{F|_{M_{g,j}}} \right] \sum_k \left( \frac{1}{4}\dim N^{\beta_k} \chi(M_{g,j}) - \frac{1}{2} \sum_{x\in B\cap M_{g,j}} (-1)^{\mathrm{ind}_g(x)} n_-(\beta_k)(x) \right) \\
&\quad \cdot \int_1^{+\infty} \left( \frac{\sinh(2t)}{\cosh(2t)-\cos(\beta_k)} - 1 \right) \frac{dt}{t} \\
&\quad + \frac{1}{2}\left( \widetilde{\chi}_g'(F) - \frac{n}{2}\chi_g(F) \right) \log(Td) + \frac{a_{-1}}{\sqrt{d}} - \sqrt{T}a_{-1} + o(1).
\end{aligned}
$$

Combining (4.4), (4.14) and (4.20)-(4.25), one deduces, by setting $T \to +\infty$,

that

$$
\begin{aligned}
(4.26)\quad \log\left(\frac{P_\infty^{\det H}\left(b^{\mathrm{RS}}_{\det(H^*(M,F),G)}\right)}{b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}}(g)\right) \simeq&\\
-2\mathrm{Tr}_s^{B_g}[f]T+\left(\widetilde{\chi}_g'(F)-\frac{n}{2}\chi_g(F)\right)\log T-\left(\widetilde{\chi}_g'(F)-\frac{n}{2}\chi_g(F)\right)\log\pi&\\
-\int_{M_g}\theta_g\left(F,b^F\right)(\nabla f)^*\psi\left(TM_g,\nabla^{TM_g}\right)&\\
+2\sqrt{T}\int_{M_g}\mathrm{Tr}_F[g]\int^B L\exp\left(-B_T\right)-2\sqrt{T}a_{-1}+2T\int_{M_g}\mathrm{Tr}_F[g]f\int^B\exp\left(-B_T\right)&\\
-2T\int_{M_g}\mathrm{Tr}_F[g]f\int^B\exp\left(-B_0\right)&\\
-\sum_j\mathrm{Tr}\left[g|_{F|_{M_{g,j}}}\right]\left\{\sum_{x\in B\cap M_{g,j}}(-1)^{\mathrm{ind}_g(x)}\mathrm{ind}_g(x)-\frac{1}{2}\sum_j\chi(M_{g,j})\mathrm{dim}M_{g,j}\right\}&\\
\cdot\left(\int_0^1\left(\frac{1+e^{-2t}}{1-e^{-2t}}-\frac{1}{t}\right)\frac{dt}{t}+\int_1^{+\infty}\frac{2e^{-2t}}{1-e^{-2t}}\frac{dt}{t}\right)&\\
+2\sum_j\mathrm{Tr}\left[g|_{F|_{M_{g,j}}}\right]\sum_k\left(\frac{1}{4}\mathrm{dim}N^{\beta_k}\chi(M_{g,j})-\frac{1}{2}\sum_{x\in B\cap M_{g,j}}(-1)^{\mathrm{ind}_g(x)}n_-(\beta_k)(x)\right)&\\
\cdot\left(\int_0^1\left(\frac{\sinh(2t)}{\cosh(2t)-\cos(\beta_k)}\right)\frac{dt}{t}+\int_1^{+\infty}\left(\frac{\sinh(2t)}{\cosh(2t)-\cos(\beta_k)}-1\right)\frac{dt}{t}\right)&\\
-\left(\widetilde{\chi}_g'(F)-\frac{n}{2}\chi_g(F)\right)\log(Td)-2\frac{a_{-1}}{\sqrt{d}}+2\sqrt{T}a_{-1}&\\
+2T\int_{M_g}\mathrm{Tr}_F[g]fe\left(TM_g,\nabla^{TM_g}\right)+\frac{2a_{-1}}{\sqrt{d}}-\left(\Gamma'(1)-\log d\right)\left(\widetilde{\chi}_g'(F)-\frac{n}{2}\chi_g(F)\right)+o(1).&
\end{aligned}
$$

By [BZ1, Theorem 3.20] and [BZ1, (7.72)], one has

$$
\begin{aligned}
(4.27)\quad &\lim_{T\to+\infty}\left(2T\int_{M_g}\mathrm{Tr}_F[g]f\int^B\exp(-B_T)-2T\mathrm{Tr}_s^{B_g}[f]\right)\\
&=-\sum_j\mathrm{Tr}\left[g|_{F|_{M_{g,j}}}\right]\left\{\sum_{x\in B\cap M_{g,j}}(-1)^{\mathrm{ind}_g(x)}\mathrm{ind}_g(x)-\frac{1}{2}\sum_j\chi(M_{g,j})\mathrm{dim}M_{g,j}\right\},
\end{aligned}
$$

$$
\begin{aligned}
(4.28)\quad &\lim_{T\to+\infty}2\sqrt{T}\int_{M_g}\mathrm{Tr}_F[g]\int^B L\exp(-B_T)\\
&=2\sum_j\mathrm{Tr}\left[g|_{F|_{M_{g,j}}}\right]\left\{\sum_{x\in B\cap M_{g,j}}(-1)^{\mathrm{ind}_g(x)}\mathrm{ind}_g(x)-\frac{1}{2}\sum_j\chi(M_{g,j})\mathrm{dim}M_{g,j}\right\}.
\end{aligned}
$$

On the other hand, by [BZ1, (7.93)] and [BZ2, (5.55)], one has

$$\int_0^1 \left(\frac{1+e^{-2t}}{1-e^{-2t}} - \frac{1}{t}\right)\frac{dt}{t} + \int_1^{+\infty} \frac{2\,e^{-2t}}{1-e^{-2t}}\frac{dt}{t} = 1 - \log\pi - \Gamma'(1), \tag{4.29}$$

$$\begin{aligned}\int_0^1 \left(\frac{\sinh(2t)}{\cosh(2t)-\cos(\beta_k)}\right)\frac{dt}{t} + \int_1^{+\infty}\left(\frac{\sinh(2t)}{\cosh(2t)-\cos(\beta_k)} - 1\right)\frac{dt}{t} \\ = -\log(\pi) - \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{\beta_k}{2\pi}\right) + \frac{\Gamma'}{\Gamma}\left(1-\frac{\beta_k}{2\pi}\right)\right).\end{aligned} \tag{4.30}$$

Also, by [BZ2, (5.64)], if $x \in B \cap M_g$,

$$\frac{\dim N^{\beta_k}}{4} - \frac{n_-(\beta_k)(x)}{2} = \frac{1}{4}\left[n_+(\beta_k)(x) - n_-(\beta_k)(x)\right]. \tag{4.31}$$

From (4.26)-(4.31), we get (3.7), which completes the proof of Theorem 3.1.

# 5 Proofs of the intermediary Theorems

The purpose of this section is to give a sketch of the proofs of the intermediary Theorems. Since the methods of the proofs of these theorems are essentially the same as the corresponding theorem in [SZ], so we will refer to [SZ] for related definitions and notations directly when there will be no confusion, such as $B_{b,g}$, $A_{b,t,T}$, $A_{g,t,T}$, $C_{t,T}$, $\cdots$.

## 5.1 Proof of Theorem 4.1

From Theorem 2.5 and [SZ, (4.44)] which in our situation we also have that $P_{\infty,T}$ commutate with $g \in G$, one finds

(5.1)

$$\frac{P_T^{[0,1],\det H}\left(b^{\mathrm{RS}}_{\det(H^*(\Omega^*_{[0,1],T}(M,F)),G)}\right)}{b^{\mathcal{M},-X}_{\det(H^*(W^u,F),G)}}(g) = \prod_{i=0}^n \det\left(P^{\#}_{\infty,T}P_{\infty,T}\Big|_{\Omega^i_{[0,1],T}(M,F)}\right)^{(-1)^{i+1}}(g).$$

From [SZ, Propositions 4.4 and 4.5], one deduces that as $T \to +\infty$,

$$\begin{aligned}(5.2)\quad \det&\left(P^{\#}_{\infty,T}P_{\infty,T}\Big|_{\Omega^i_{[0,1],T}(M,F)}\right)(g) \\
&= \det\left(e_T e_T^{\#}P^{\#}_{\infty,T}P_{\infty,T}\Big|_{\Omega^i_{[0,1],T}(M,F)}\right)(g)\cdot\det{}^{-1}\left(e_Te_T^{\#}\Big|_{\Omega^i_{[0,1],T}(M,F)}\right)(g) \\
&= \det\left(\left(P_{\infty,T}e_T\right)^{\#}P_{\infty,T}e_T\Big|_{C^i(W^u,F)}\right)(g)\cdot\det{}^{-1}\left(e_T^{\#}e_T\Big|_{C^i(W^u,F)}\right)(g) \\
&= \det\left(\left(1+O\left(e^{-cT}\right)\right)^{\#}\left(\frac{\pi}{T}\right)^{N-n/2}e^{2T\mathcal{F}}\left(1+O\left(e^{-cT}\right)\right)\Big|_{C^i(W^u,F)}\right)(g) \\
&\qquad\cdot\det{}^{-1}\left(\left(1+O\left(e^{-cT}\right)\right)\Big|_{C^i(W^u,F)}\right)(g).\end{aligned}$$

From (5.1) and (5.2), one gets (4.4) immediately.

The proof of Theorem 4.1 is completed. Q.E.D.

### 5.2 Proof of Theorem 4.2

The proof of Theorem 4.2 is the same as the proof of [SZ, Theorem 3.4] given in [SZ, Section 5].

### 5.3 Proof of Theorem 4.3

Recall that the operator $e_T : C^*(W^u, F) \to \Omega^*_{[0,1],T}(M, F)$ has been defined in [SZ, (4.38)], and in the current case, we also have that $e_T$ commute with $G$. So by [SZ, Proposition 4.4], we have that for $T \geq 0$ large enough, $e_T : C^*(W^u, F) \to \Omega^*_{[0,1],T}(M, F)$ is an identification of $G$-spaces. So (4.7) follows. Also (4.8) was already proved in [SZ, Theorem 3.5].

### 5.4 Proof of Theorem 4.4

In this section, we provide a proof of Theorem 4.4, which computes the asymptotic of $\mathrm{Tr}_s[gN\exp(-tD^2_{b_T})]$ for fixed $T \geq 0$ as $t \to 0$. The method is the essentially same as it in [SZ].

By [SZ, (6.4)], we have

(5.3)
$$\begin{aligned} e^{-tD_b^2} = e^{-tD_g^2} &+ \sum_{k=1}^{n}(-1)^k t^k \int_{\Delta_k} e^{-t_1tD_g^2}B_{b,g}e^{-t_2tD_g^2}\cdots B_{b,g}e^{-t_{k+1}tD_g^2}dt_1\cdots dt_k \\ &+ (-1)^{n+1}t^{n+1}\int_{\Delta_{n+1}} e^{-t_1tD_g^2}B_{b,g}e^{-t_2tD_g^2}\cdots B_{b,g}e^{-t_{n+2}tD_b^2}dt_1\cdots dt_{n+1}, \end{aligned}$$

where $\Delta_k$, $1 \leq k \leq n+1$, is the $k$-simplex defined by $t_1 + \cdots + t_{k+1} = 1$, $t_1 \geq 0$, $\cdots$, $t_{k+1} \geq 0$. Also, by the same proof of [SZ, Proposition 6.1], we have the following result.

**Proposition 5.1.** *As $t \to 0^+$, one has*

(5.4)
$$t^{n+1}\int_{\Delta_{n+1}} \mathrm{Tr}_s\left[gNe^{-t_1tD_g^2}B_{b,g}e^{-t_2tD_g^2}\cdots B_{b,g}e^{-t_{n+2}tD_b^2}\right]dt_1\cdots dt_{n+1} \to 0.$$

By [SZ, (6.22) and (6.23)], we have that for any $1 < k \leq n$, $(t_1, \cdots, t_{k+1}) \in \Delta_k$,

$$\lim_{t\to 0^+} t^k \mathrm{Tr}_s\left[gNe^{-t_1tD_g^2}B_{b,g}e^{-t_2tD_g^2}\cdots B_{b,g}e^{-t_{k+1}tD_g^2}\right] = 0, \tag{5.5}$$

while for $k=1, 0 \leq t_1 \leq 1$,

$$\text{(5.6)} \quad \lim_{t\to 0^+} t\mathrm{Tr}_s\left[gNe^{-t_1tD_g^2}B_{b,g}e^{-(1-t_1)tD_g^2}\right] = \lim_{t\to 0^+} t\mathrm{Tr}_s\left[gNB_{b,g}e^{-tD_g^2}\right]$$
$$= \frac{1}{2}\int_{M_g}\int^B \mathrm{Tr}\left[g\left(\sum_{i,\,j=1}^n e_i\wedge \widehat{e_j}\left(\nabla^u_{e_i}\omega^F\left(e_j\right)\right) + \frac{1}{2}\left[\omega^F, \widehat{\omega^F_g} - \widehat{\omega^F}\right]\right)\right]$$
$$\cdot L\exp\left(-\frac{\dot{R}^{TM_g}}{2}\right).$$

So by [BZ2, (2.13)], and proceed as in [SZ, (6.26)-(6.28)], we have

$$\text{(5.7)} \qquad \lim_{t\to 0^+} t\mathrm{Tr}_s\left[gNe^{-t_1tD_g^2}B_{b,g}e^{-(1-t_1)tD_g^2}\right] = 0.$$

From (5.3), (5.4), (5.5), (5.7) and [BZ2, Theorem 5.9], one gets (4.9). The proof of Theorem 4.4 is completed. Q.E.D.

## 5.5 Proof of Theorem 4.5

In order to prove (4.10), one need only to prove that under the conditions of Theorem 4.5, there exists constant $C'' > 0$ such that

(5.8)
$$\left|\mathrm{Tr}_s\left[gN\exp\left(-\left(tD_b + T\widehat{c}(\nabla f)\right)^2\right)\right] - \mathrm{Tr}_s\left[gN\exp\left(-\left(tD_g + T\widehat{c}(\nabla f)\right)^2\right)\right]\right.$$
$$\left. -\frac{T}{2}\int_{M_g}\left(\theta_g\left(F,b^F\right) - \theta_g\left(F,g^F\right)\right)\int^B \widehat{df}\exp\left(-B_{T^2}\right)\right| \leq C''t.$$

By [SZ, (7.8)], we have

$$\text{(5.9)} \quad e^{-A^2_{b,t,T}} = e^{-A^2_{g,t,T}}$$
$$+\sum_{k=1}^n(-1)^k\int_{\Delta_k} e^{-t_1A^2_{g,t,T}}C_{t,T}e^{-t_2A^2_{g,t,T}}\cdots C_{t,T}e^{-t_{k+1}A^2_{g,t,T}}dt_1\cdots dt_k$$
$$+(-1)^{n+1}\int_{\Delta_{n+1}} e^{-t_1A^2_{g,t,T}}C_{t,T}e^{-t_2A^2_{g,t,T}}\cdots C_{t,T}e^{-t_{n+2}A^2_{b,t,T}}dt_1\cdots dt_{n+1}.$$

By the same proof of [SZ, (7.21)], we have that there exists $C_1 > 0$ such that for any $t > 0$ small enough and $T \in [0, \frac{1}{t}]$,

(5.10)
$$\left|\int_{\Delta_{n+1}} \mathrm{Tr}_s\left[gNe^{-t_1A^2_{g,t,T}}C_{t,T}e^{-t_2A^2_{g,t,T}}\cdots C_{t,T}e^{-t_{n+2}A^2_{b,t,T}}\right]dt_1\cdots dt_{n+1}\right| \leq C_1t.$$

Also by the same proof of [SZ, (7.23)], we have that there exists $C_2 > 0$, $0 < d < 1$ such that for any $1 < k \leq n$, $0 < t \leq d$, $T \geq 0$ with $tT \leq 1$,

(5.11)
$$\left|\int_{\Delta_k} \mathrm{Tr}_s\left[gNe^{-t_1A^2_{g,t,T}}C_{t,T}e^{-t_2A^2_{g,t,T}}\cdots C_{t,T}e^{-t_{k+1}A^2_{g,t,T}}\right]dt_1\cdots dt_k\right| \leq C_2t,$$

while for $k=1$ one has for any $0<t\leq d$, $T\geq 0$ with $tT\leq 1$ and $0\leq t_1\leq 1$, by [BZ2, Proposition 9.3], we have

(5.12)
$$\left|\mathrm{Tr}_s\left[gNe^{-t_1A^2_{g,t,T}}C_{t,T}e^{-(1-t_1)A^2_{g,t,T}}\right]-T\int_{M_g}\int^B\mathrm{Tr}\left[g\omega^F(\nabla f)\right]L\exp\left(-B_{T^2}\right)\right| \leq C_2t.$$

Now similar as [SZ, (7.25)], we have

$$(5.13)\quad \int_{M_g}\int^B\mathrm{Tr}\left[g\omega^F(\nabla f)\right]L\exp\left(-B_{T^2}\right) = \frac{1}{2}\int_{M_g}\left(\theta_g\left(F,g^F\right)-\theta_g\left(F,b^F\right)\right)\int^B\widehat{\nabla f}\exp\left(-B_{T^2}\right).$$

From (5.9)-(5.13), we get (5.8), which completes the proof of Theorem 4.5. Q.E.D.

## 5.6 Proof of Theorem 4.6

In order to prove Theorem 4.6, we need only to prove that for any $T>0$,

$$(5.14)\quad \lim_{t\to 0^+}\left(\mathrm{Tr}_s\left[gN\exp\left(-A^2_{b,t,\frac{T}{t}}\right)\right]-\mathrm{Tr}_s\left[gN\exp\left(-A^2_{g,t,\frac{T}{t}}\right)\right]\right)=0.$$

By [SZ, (8.2) and (8.4)], there exists $0<C_0\leq 1$, such that when $0<t\leq C_0$, one has the absolute convergent expansion formula

$$(5.15)\quad e^{-A^2_{b,t,\frac{T}{t}}}-e^{-A^2_{g,t,\frac{T}{t}}} = \sum_{k=1}^{+\infty}(-1)^k\int_{\Delta_k}e^{-t_1A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}e^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}e^{-t_{k+1}A^2_{g,t,\frac{T}{t}}}dt_1\cdots dt_k,$$

and that

$$(5.16)\quad \sum_{k=n}^{+\infty}(-1)^k\int_{\Delta_k}ge^{-t_1A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}e^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}e^{-t_{k+1}A^2_{g,t,\frac{T}{t}}}dt_1\cdots dt_k$$

is uniformly absolute convergent for $0<t\leq C_0$.

Proceed as in [SZ, Section 8], one has that for any $(t_1,\cdots,t_{k+1})\in\Delta_k\setminus\{t_1\cdots t_{k+1}=0\}$,

$$(5.17)\quad \left|\mathrm{Tr}_s\left[gNe^{-t_1A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}e^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}e^{-t_{k+1}A^2_{g,t,\frac{T}{t}}}\right]\right| \leq C_3t^k\left(t_1\cdots t_k\right)^{-\frac{1}{2}}\mathrm{Tr}\left[e^{-\frac{A^2_{g,t,\frac{T}{t}}}{2}}\right]\left\|\psi e^{-\frac{t_{k+1}}{2}A^2_{g,t,\frac{T}{t}}}\right\|$$

for some positive constant $C_3 > 0$.

Also, by [SZ, (8.4)], (5.17) and the same assumption in [SZ] that $t_{k+1} \geq \frac{1}{k+1}$, one gets

$$\left|\int_{\Delta_k} \mathrm{Tr}_s\left[gNe^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} e^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} e^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}\right] dt_1 \cdots dt_k\right| \leq C_4 t^{k-n} \left\|\psi e^{-\frac{1}{2(k+1)} A^2_{g,t,\frac{T}{t}}}\right\| \tag{5.18}$$

for some constant $C_4 > 0$.

From (5.15), (5.16), (5.18), [SZ, (8.9) and (8.10)] and the dominate convergence, we get (5.14), which completes the proof of Theorem 4.6. Q.E.D.

## 5.7 Proof of Theorem 4.7

In order to prove Theorem 4.7, we need only to prove that there exist $c > 0$, $C > 0$, $0 < C_0 \leq 1$ such that for any $0 < t \leq C_0$, $T \geq 1$,

$$\left|\mathrm{Tr}_s\left[gN\exp\left(-A^2_{b,t,\frac{T}{t}}\right)\right] - \mathrm{Tr}_s\left[gN\exp\left(-A^2_{g,t,\frac{T}{t}}\right)\right]\right| \leq c\exp(-CT). \tag{5.19}$$

First of all, one can choose $C_0 > 0$ small enough so that for any $0 < t \leq C_0$, $T > 0$, by (5.15), we have the absolute convergent expansion formula

$$\begin{aligned} & e^{-A^2_{b,t,\frac{T}{t}}} - e^{-A^2_{g,t,\frac{T}{t}}} \\ & = \sum_{k=1}^{+\infty}(-1)^k \int_{\Delta_k} e^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} e^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} e^{-t_{k+1} A^2_{g,t,\frac{T}{t}}} dt_1 \cdots dt_k, \end{aligned} \tag{5.20}$$

from which one has

$$\begin{aligned} & \mathrm{Tr}_s\left[gN\exp\left(-A^2_{b,t,\frac{T}{t}}\right)\right] - \mathrm{Tr}_s\left[gN\exp\left(-A^2_{g,t,\frac{T}{t}}\right)\right] \\ & = \sum_{k=1}^{+\infty}(-1)^k \int_{\Delta_k} \mathrm{Tr}_s\left[gNe^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} e^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} e^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}\right] dt_1 \cdots dt_k. \end{aligned} \tag{5.21}$$

Thus, in order to prove (5.19), we need only to prove

$$\begin{aligned} & \sum_{k=1}^{+\infty}\left|\int_{\Delta_k} \mathrm{Tr}_s\left[gNe^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} e^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} e^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}\right] dt_1 \cdots dt_k\right| \\ & = \sum_{k=1}^{+\infty}\left|\int_{\Delta_k} \mathrm{Tr}_s\left[gNe^{-(t_1+t_{k+1}) A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} e^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}}\right] dt_1 \cdots dt_k\right| \\ & \leq c\exp(-CT). \end{aligned} \tag{5.22}$$

By [SZ, (8.6)], we have for any $t > 0$, $T \geq 1$, $(t_1, \cdots, t_{k+1}) \in \Delta_k \setminus \{t_1 \cdots t_{k+1} = 0\}$,

$$\begin{aligned}(5.23)\quad & \mathrm{Tr}_s\left[gNe^{-(t_1+t_{k+1})A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}e^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}\right] \\ & = \mathrm{Tr}_s\left[gN\psi e^{-(t_1+t_{k+1})A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\psi e^{-t_2A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\cdots \psi e^{-t_kA^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\right].\end{aligned}$$

From (5.23), [SZ, (9.18) and (9.19)], one sees that there exists $C_5 > 0$, $C_6 > 0$ and $C_7 > 0$ such that for any $k \geq 1$,

$$\begin{aligned}(5.24)\quad & \left|\int_{\Delta_k}\mathrm{Tr}_s\left[gNe^{-t_1A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}e^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}e^{-t_{k+1}A^2_{g,t,\frac{T}{t}}}\right]dt_1\cdots dt_k\right| \\ & \leq C_5\left(C_6t\right)^k\frac{T^{\frac{n}{2}}}{t^n}\exp\left(-\frac{C_7T}{4}\right),\end{aligned}$$

from which one sees that there exists $0 < c_1 \leq 1$, $C_8 > 0$, $C_9 > 0$ such that for any $0 < t \leq c_1$ and $T \geq 1$, one has

$$\begin{aligned}(5.25)\quad & \left|\sum_{k=n}^{+\infty}\int_{\Delta_k}\mathrm{Tr}_s\left[gNe^{-t_1A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}e^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}e^{-t_{k+1}A^2_{g,t,\frac{T}{t}}}\right]dt_1\cdots dt_k\right| \\ & \leq C_8\exp\left(-C_9T\right).\end{aligned}$$

On the other hand, for any $1 \leq k < n$, by proceeding as in (5.18), one has that for any $0 < t \leq c_1$, $T \geq 1$,

$$\begin{aligned}(5.26)\quad & \left|\int_{\Delta_k}\mathrm{Tr}_s\left[gNe^{-t_1A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}e^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}e^{-t_{k+1}A^2_{g,t,\frac{T}{t}}}\right]dt_1\cdots dt_k\right| \\ & \leq C_{10}t^{k-n}\left\|\psi e^{-\frac{1}{2(k+1)}A^2_{g,t,\frac{T}{t}}}\right\|\end{aligned}$$

for some constant $C_{10} > 0$.

From (5.26) and [SZ, (9.23)], one sees immediately that there exists $C_{11} > 0$, $C_{12} > 0$ such that for any $1 \leq k \leq n-1$, $0 < t \leq c_1$ and $T \geq 1$, one has

$$\begin{aligned}(5.27)\quad & \left|\int_{\Delta_k}\mathrm{Tr}_s\left[gNe^{-t_1A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}e^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}e^{-t_{k+1}A^2_{g,t,\frac{T}{t}}}\right]dt_1\cdots dt_k\right| \\ & \leq C_{11}e^{-C_{12}T}.\end{aligned}$$

From (5.21), (5.25) and (5.27), one gets (5.19).

The proof of Theorem 4.7 is completed. Q.E.D.

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